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Sectoral Income

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Abstract What is the income of each sector of an economy? E.g., in the case of a country endowed with petroleum resources, what is the income of its petroleum sector? Here we present a definition of sectoral income, which is compatible with an important line of theoretical literature on comprehensive national accounting. We do so first by splitting national income into individual income and then defining sectoral income by considering the contributions to individual income that the sectors give rise to.

Keywords Sectoral income · Comprehensive national accounting

JEL Classification C43 · D60 · O47

1 Introduction

What is the income of each sector of an economy? E.g., in the case of a country endowed with petroleum resources, what is the income of its petroleum sector?

In practical applications, sectoral income has often been measured by wealth-based measures, whereby the present value of the cash flow from a sector is estimated, and its income is equated with the interest on the sector's wealth determined in this manner (see, e.g., Aslaksen et al. 1990, and Brekke 1997, Sects. II.C and IV).

In contrast, a line of theoretical literature—from Hicks (1946, Chap. 14) via Samuelson (1961) to Sefton and Weale (2006)—has taken a quite different route by associating income with the present value of real interest on future consumption and savings with the present value of future consumption changes.

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In this paper we extend this line of theoretical literature on comprehensive national accounting by presenting a definition of sectoral income which is consistent with such consumption-based income definitions. To be precise, we define sectoral income by keeping track of how each sector contributes to individual income (in the sense of Sefton and Weale, 2006, Definition 2).

We start in Sect. 2 by presenting a short survey of relevant literature and the income concepts presented in these contributions. Then, in Sects. 3–5 we present definitions of real income at the national and individual level in line with Sefton and Weale (2006), while generalizing their welfare results slightly. Our definition of sectoral income is presented and analyzed in Sect. 6, before illustrating this concept both in the setting of a general model (in Sect. 7) and in the setting of a partial model (in Sect. 8). Throughout we seek to derive expressions for sectoral income that can be useful in practical applications.

There are two appendices, one contains welfare results referred to in the main part of our paper, while the other analyzes alternative wealth-based concepts of sectoral income.

2 What is Income?

At a national level, particularly in the context of a closed economy with a stationary technology, income can be derived from net national product, measuring the value of the flows of goods and services that are produced by the productive assets of society. National income derived in this way has also welfare significance, as established by Weitzman (1976) and later references (e.g., Aronsson et al. 1997; Aronsson et al. 2004; Asheim and Weitzman 2001). At a sectoral level, it is however hard to determine what a sector's "net product" is, since much of the return on the sector's assets may derive from expected capital gains.[1] In particular, the remaining deposits of a non-renewable resource is not productive as a stock, but yields its owners positive returns by being moved closer to the time of depletion. This motivates a brief survey of relevant literature on income concepts.

Income in the tradition of Fisher (1906) and Lindahl (1933, Sect. II) is associated with interest on wealth, where wealth is the present value of future consumption. If, at each point in time, national consumption equals the sum of the cash flows from the different sectors of the economy, this definition allows national income to be split into sectoral income so that sectoral income summed over all sectors adds up to national income.

Hicks (1946), in Chapter 14 of *Value and Capital*, suggests that "the practical purpose of income is to serve as a guide for prudent conduct" by giving "people an indication of the amount which they can consume without impoverish themselves" (both quotes from Hicks 1946, p. 172).

Hicks (1946, p. 174) points out that income as interest on wealth is not an indicator of prudent behavior if the real interest rate is expected to change. This observation is nicely illustrated by the Dasgupta-Heal-Solow model (Dasgupta and Heal 1974, 1979; Solow 1974) of capital accumulation and resource depletion—which we will return to in Sect. 7—where the real interest rate is decreasing along a path where capital is accumulated and resource flow diminishes. In this model, income as interest on wealth exceeds both net national product and consumption along an efficient path with constant consumption (see Appendix B for details). Hence, in this setting, the consumers of the economy would impoverish themselves if they were consuming the interest on their wealth.

[1] See, however, an interesting attempt to do so in Sefton and Weale (2006, Sect. 6.2).

If we instead use Hicks's (1946) suggestion to obtain alternative income concepts, then we must operationalize what is meant by "the amount which they can consume without impoverish themselves". Hicks (1946) himself offers the following operationalization, referring to the corresponding concept as "Income No. 3":

> "Income No. 3 must be defined as the maximum amount of money which the individual can spend this week, and still be able to spend the same amount *in real terms* in each ensuing week" (Hicks 1946, p. 174). "The standard stream corresponding to Income No. 3 is constant in real terms .... We ask ... how much he would be receiving if he were getting a standard stream of the same present value as his actual expected receipts. This amount is his income. (Hicks 1946, p. 184)"

Hence, income is associated with the "*stationary equivalent* of future consumption" (Weitzman 1976, p. 160).

In an economy where well-being depends on a single consumption good, this concept of income can be defined as the constant level of consumption with the same present value as the actual future stream of consumption. Such *wealth equivalent income* can be determined at both a national and sectoral level in such way that sectoral income summed over all sectors adds up to national income (see Appendix B). Moreover, the concept is designed to be an indicator of prudent behavior (although wealth equivalent income is only hypothetically sustainable if interest rates are changed when consumption is transformed into a constant and efficient path).

Unfortunately, as pointed out by Asheim (1997) and Sefton and Weale (2006, Sect. 3.1.2) (and discussed in Appendix B), such wealth equivalent income at the national level does not equal net national product, even in a closed economy with a stationary technology, unless the interest rate is constant (which is the case analyzed by Weitzman 1976) or the consumption level is constant (in which case Hartwick's rule[2] implies that net national product equals this constant level and, thus, the result follows). Moreover, this concept is hard to generalize to the empirically relevant case of multiple consumption goods, since determining an amount constant in real terms leads to an indexing problem if relative consumption prices are changing.

However, Hicks's "amount which they can consume without impoverish themselves" can be interpreted in an alternative manner. Hicks (1946, p. 172) writes that "it seems that we ought to define a man's income as the maximum value which he can consume during a week, and still expect to be as well off at the end of the week as he was at the beginning." One attractive possibility, suggested by Pemberton and Ulph (2001) and Sefton and Weale (2006), is to associate "as well off" with the level of dynamic welfare.

It is an insight first pointed out by Samuelson (1961, pp. 51–52) that the present value of future consumption changes measures welfare improvement in a market economy following an optimal path. This gives a welfare foundation for interpreting the present value of future consumption changes as national savings. Adding current consumption to this notion of savings (measured in the same numeraire) leads to a concept of national income with nice properties:

(1) It follows from Samuelson's insight that such a concept of national income is an indicator of prudent behavior, since the present value of future consumption changes is positive—and thus, dynamic welfare improves—if and only if consumption is smaller than national income.

[2] Cf. Hartwick (1977) and Dixit et al. (1980).

(2) It follows through integration by parts that such a concept of national income can be expressed as the present value of real interest on future national consumption.
(3) It follows from the analysis of Sefton and Weale (1996) and Weitzman (2003, Chap. 6) that such a concept of national income equals net national product in a closed economy with a stationary technology.

In Sects. 3 and 4, and backed up by the results of Appendix A, we establish formally properties (1)–(3) under assumptions more general than those imposed by Sefton and Weale (2006); in particular, we do not assume that a discounted utilitarian welfare function is maximized, and we do not require that the technology satisfies constant-returns-to-scale. In Sects. 5 and 6 we then turn to the up-to-now unresolved question of how to split this concept of national income into sectoral income in such way that sectoral income summed over all sectors adds up to national income. We do so first, in Sect. 5, by splitting national income into individual income, building on analysis presented by Sefton and Weale (2006), and then, in Sect. 6, by defining sectoral income by considering the contributions to individual income that the sectors give rise to. Throughout (and in line with the analysis of Sefton and Weale 2006), consumer price indices play a central and natural role when turning nominal into real prices.

3 Defining National Income

Consider a national economy, where $\mathbf{c}$ is a comprehensive vector of consumption flows, implying that all determinants of current well-being are included in $\mathbf{c}$. Let $\{\mathbf{c}(t)\}_{t=0}^{\infty}$ be the path of consumption flows in this economy, and let $\{\mathbf{p}_c(t)\}_{t=0}^{\infty}$ be the corresponding path of market (or calculated) *present value* prices of consumption. The term "present value" reflects that discounting is taken care of by the prices. In particular, if relative consumption prices are constant throughout and there is constant real interest rate R, then it holds that $\mathbf{p}_c(t) = e^{-Rt}\mathbf{p}_c(0)$. However, we will allow for non-constant relative consumption prices and will return to the question of how to determine real interest rates from $\{\mathbf{p}_c(t)\}_{t=0}^{\infty}$ in this more general case.

Differentiation of $\mathbf{p}_c(t)\mathbf{c}(t)$ yields

$$\frac{d}{dt}\left(\mathbf{p}_c(t)\mathbf{c}(t)\right) = \dot{\mathbf{p}}_c(t)\mathbf{c}(t) + \mathbf{p}_c(t)\dot{\mathbf{c}}(t).$$

Integrating on both sides under the assumption that $\mathbf{p}_c(\tau)\mathbf{c}(\tau) \to 0$ as $\tau \to \infty$, leads to the following equation:

$$-\mathbf{p}_c(t)\mathbf{c}(t) = \int_t^{\infty} \dot{\mathbf{p}}_c(\tau)\mathbf{c}(\tau)d\tau + \int_t^{\infty} \mathbf{p}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau.$$

By rearranging this equality we obtain

$$\underbrace{\int_t^{\infty} (-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}(\tau)d\tau}_{\text{National income}} = \mathbf{p}_c(t)\mathbf{c}(t) + \underbrace{\int_t^{\infty} \mathbf{p}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau}_{\text{National savings}}. \quad (1)$$

Here, we will interpret the l.h.s. as *national income* at time t and the second term on the r.h.s. as *national savings* at time t. As we will argue next, these interpretations can be supported in both a welfare and a productive perspective.

In line with Samuelson (1961, pp. 51–52), one can argue that $\int_t^\infty \mathbf{p}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau$ measures welfare improvement in a market economy following an optimal path. A precise and more general statement of this result is proven in Appendix A through Proposition 4. In particular, we need not assume that the dynamic welfare is discounted utilitarian. Moreover, by allowing for the possibility that the prices are calculated, we need not assume that the economy implements a welfare maximizing path of consumption flows through an intertemporal market equilibrium.

Thus, Proposition 4 gives a welfare foundation for interpreting $\int_t^\infty \mathbf{p}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau$ as national savings. Then, if national income is to serve as a guide for prudent conduct in the sense that dynamic welfare improves if and only if national consumption is smaller than national income, we obtain that national income equals $\mathbf{p}_c(t)\mathbf{c}(t) + \int_t^\infty \mathbf{p}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau$, which by (1) can be transformed to $\int_t^\infty (-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}(\tau)d\tau$.

If an economy implements a path with constant instantaneous well-being and the vector of consumption prices $\mathbf{p}_c(t)$ is at any time proportional to the contributions that the various consumption flows make to instantaneous well-being, then it follows that $\mathbf{p}_c(t)\dot{\mathbf{c}}(t) = 0$ at all times. Hence, national income equals the value of consumption and shows that this concept of income serves as a guide for prudent conduct also in this special case.

Under the assumptions of the technology being stationary and the economy realizing a competitive equilibrium, then it follows from Dixit et al. (1980, proof of Theorem 1) that

$$\mathbf{p}_c(t)\dot{\mathbf{c}}(t) + \frac{d}{dt}\left(\mathbf{p}_k(t)\dot{\mathbf{k}}(t)\right) = 0,$$

where $\{\mathbf{k}(t)\}_{t=0}^\infty$ is the path of the vector of capital stocks in this economy and $\{\mathbf{p}_k(t)\}_{t=0}^\infty$ is the corresponding path of market (or calculated) present value prices of net investment flows.

Integrating on both sides under the assumption that $\mathbf{p}_k(\tau)\dot{\mathbf{k}}(\tau) \to 0$ as $\tau \to \infty$, entails that the following equation holds for all t:

$$\int_t^\infty \mathbf{p}_c(\tau)\dot{\mathbf{c}}(\tau)dt = \mathbf{p}_k(t)\dot{\mathbf{k}}(t).$$

Combined with (1) we obtain:

$$\underbrace{\int_t^\infty (-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}(\tau)d\tau}_{\text{National income}} = \underbrace{\mathbf{p}_c(t)\mathbf{c}(t) + \mathbf{p}_k(t)\dot{\mathbf{k}}(t)}_{\text{Net national product}}. \tag{2}$$

Hence, national income as defined through (1) equals net national product under the assumptions of the technology being stationary and the economy realizing a competitive equilibrium.

A precise and more general statement of the result that the value of the net investment flows equals the present value of future consumption changes is proven in Appendix A through Proposition 5. In particular, we need not assume that the economy implements a competitive equilibrium. By allowing for the possibility that the prices are calculated, it is sufficient that the path of consumption flows and capital stocks is implemented by a stationary *resource allocation mechanism* (as introduced by Dasgupta and Mäler 2000; Dasgupta 2001; Arrow et al. 2003).

Example Cake-eating economy. It is instructive to illustrate this definition of national income in the setting of a cake-eating economy, faced with the problem

$$\max_{\{c(t)\}_{t=0}^{\infty}} \int_0^{\infty} e^{-\rho t} u(c(t))dt \quad \text{s.t.} \quad \int_0^{\infty} c(t)dt \leq S(0) \;\; \text{and} \;\; c(t) \geq 0 \quad \text{for all } t \geq 0$$

for some twice differentiable and strictly concave utility function $u : [0, \infty) \to \mathbb{R}$ satisfying $\lim_{c \to 0} u'(c) = \infty$, utility discount rate $\rho > 0$ and initial cake $S(0) > 0$. The optimal path, $\{c(t)\}_{t=0}^{\infty}$, is differentiable and satisfies

$$p_c(t) := e^{-\rho t} u'(c(t)) = u'(c(0)) \quad \text{for all } t \geq 0.$$

Since $\dot{p}_c(t) = 0$ for all $t \geq 0$, national income at each time t equals zero:

$$\int_t^{\infty} (-\dot{p}_c(\tau))\, c(\tau) d\tau = 0.$$

Moreover, it follows from (1) that the positive value of consumption at each time t exactly cancels the negative present value of the future consumption changes, the latter term measuring the change in dynamic welfare as the remaining cake vanishes:

$$p_c(t)c(t) + \int_t^{\infty} p_c(\tau)\dot{c}(\tau)d\tau = p_c(t)c(t) + p_c(t) \int_t^{\infty} \dot{c}(\tau)d\tau = p(t)\,(c(t) - c(t)) = 0$$

since $p_c(\tau) = p_c(t)$ for all $\tau \geq t$ and $\lim_{\tau \to \infty} c(\tau) = 0$.

4 Expressions for Real National Income

To find real (rather than present value) prices, consider the Divisia consumer price index $\{\pi(t)\}_{t=0}^{\infty}$ defined by $\pi(0) = 1$ and

$$\frac{\dot{\pi}(t)}{\pi(t)} = \frac{\dot{\mathbf{p}}_c(t)\mathbf{c}(t)}{\mathbf{p}_c(t)\mathbf{c}(t)}, \tag{3}$$

for all $t \geq 0$. Define the path of market (or calculated) *real* prices of consumption $\{\mathbf{P}_c(t)\}_{t=0}^{\infty}$ by

$$\mathbf{P}_c(t) = \mathbf{p}_c(t)/\pi(t) \tag{4}$$

for all $t \geq 0$. Define the path of market (or calculated) *real* consumption interest rates $\{R(t)\}_{t=0}^{\infty}$ by

$$R(t) = -\dot{\pi}(t)/\pi(t) \tag{5}$$

for all $t \geq 0$. Then, by applying (3)–(5),

$$(-\dot{\mathbf{p}}_c(t))\,\mathbf{c}(t) = -\frac{\dot{\pi}(t)}{\pi(t)}\mathbf{p}_c(t)\mathbf{c}(t) = \pi(t)R(t)\mathbf{P}_c(t)\mathbf{c}(t). \tag{6}$$

Hence, it follows from (1) that *real* national income, $\int_t^{\infty} (-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}(\tau)d\tau/\pi(t)$, is equal to the present value of real interest on future national consumption, as stated in the following definition.

Definition 1 *Real national income* at time t is determined as

$$Y(t) := \int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)\mathbf{P}_c(\tau)\mathbf{c}(\tau)d\tau.$$

By using (4) in (1), we can express real national income as the sum of current real national consumption and the real national savings, as stated in Proposition 1 below. Furthermore, by differentiating $Y(t)$ w.r.t. t, we obtain as the second part of the proposition that $\dot{Y}(t) \geq 0$ is equivalent to $\mathbf{P}_c(t)\mathbf{c}(t) \leq Y(t)$ and $\int_t^\infty (\pi(\tau)/\pi(t))\, \mathbf{P}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau \geq 0$ if the real interest rate $R(t)$ is positive; hence, $\dot{Y}(t) \geq 0$ can serve as an alternative guide for prudent behavior.

Proposition 1 Real national income *at time t can be expressed as*

$$Y(t) = \mathbf{P}_c(t)\mathbf{c}(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \mathbf{P}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau.$$

Furthermore,

$$\dot{Y}(t) = R(t)\left(Y(t) - \mathbf{P}_c(t)\mathbf{c}(t)\right) = R(t)\left(\int_t^\infty \frac{\pi(\tau)}{\pi(t)} \mathbf{P}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau\right).$$

Example Cake-eating economy (continued). In the case of the cake-eating economy introduced in Sect. 3, $p_c(t) = p_c(0)$ for all $t \geq 0$. It follows from $\pi(0) = 1$ and (3) that $\pi(t) = 1$ for all $t \geq 0$. Furthermore, by (5), $R(t) = 0$ for all $t \geq 0$. Hence, by applying Definition 1, we obtain that real national income, $Y(t)$, in a cake-eating economy equals zero for all t. Furthermore, since the real interest rate equals zero for all t, $\dot{Y}(t) \geq 0$ cannot serve as a guide for prudent behavior. This is caused by the fact that a cake-eating economy has only one asset, the "cake", which is unproductive as a stock. In the respect, a cake-eating economy represents an extreme case, which corresponds neither to the models that economists usually analyze nor to real economies. In the Dasgupta-Heal-Solow model analyzed in Sect. 7, we combine a non-renewable resource, being unproductive as a stock, with a productive asset. This leads to a real interest rate which is positive throughout.

Definition 1 and Proposition 1 yield expressions for income that can be used at a national level also if the technology is not stationary, and it also facilitates the definition and expression of income for individuals and in different sectors of a national economy. We turn to such definitions next.

5 Defining Individual Income

Divide the national economy into m infinitely lived *individuals* (or dynasties), so that each individual i is in $I := \{1, \ldots, m\}$. For each $i \in I$, denote by $\{\mathbf{c}_i(t)\}_{t=0}^\infty$ the path of the vector of individual i's consumption flows, where

$$\mathbf{c}(t) = \sum_{i \in I} \mathbf{c}_i(t) \quad \text{for all } t \geq 0. \tag{7}$$

In line with Sefton and Weale (2006, Definition 2) and the analysis of Sect. 3, define individual income by $\int_t^\infty (-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}_i(\tau)d\tau$, which can be transformed to

$$\underbrace{\int_t^\infty (-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}_i(\tau)d\tau}_{\text{individual income}} = \mathbf{p}_c(t)\mathbf{c}_i(t) + \underbrace{\int_t^\infty \mathbf{p}_c(\tau)\dot{\mathbf{c}}_i(\tau)d\tau}_{\text{individual savings}} \tag{8}$$

through integration by parts, provided that $\mathbf{p}_c(\tau)\mathbf{c}_i(\tau) \to 0$ as $\tau \to \infty$. It follows from Proposition 6 of Appendix A that $\int_t^\infty \mathbf{p}_c(\tau)\dot{\mathbf{c}}_i(\tau)d\tau$ measures individual welfare improvement if the individual faces $\{\mathbf{p}_c(t)\}_{t=0}^\infty$ as market prices and maximizes dynamic welfare subject to a budget constraint. This result does not rely on individual welfare being discounted utilitarian. Hence, individual income as defined above serves as a guide for prudent conduct under general conditions.

To define *real* individual income at time t in the same numeraire as real national income at time t, and to obtain alternative expressions, consider individual Divisia consumer price indices $\{\pi_i(\tau)\}_{\tau=t}^\infty$ defined by $\pi_i(t) = \pi(t)$ and

$$\frac{\dot{\pi}_i(\tau)}{\pi_i(\tau)} = \frac{\dot{\mathbf{p}}_c(\tau)\mathbf{c}_i(\tau)}{\mathbf{p}_c(\tau)\mathbf{c}_i(\tau)}, \tag{9}$$

for all $\tau \geq t$. Define the path of market (or calculated) *real* prices of consumption $\{\mathbf{P}_{ci}(\tau)\}_{\tau=t}^\infty$ for individual i by

$$\mathbf{P}_{ci}(\tau) = \mathbf{p}_c(\tau)/\pi_i(\tau) \tag{10}$$

for all $\tau \geq t$. Define the path of market (or calculated) *real* consumption interest rates $\{R_i(\tau)\}_{\tau=t}^\infty$ for individual i by

$$R_i(\tau) = -\dot{\pi}_i(\tau)/\pi_i(\tau) \tag{11}$$

for all $\tau \geq t$. Then, by applying (9)–(11),

$$(-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}_i(\tau) = -\frac{\dot{\pi}_i(\tau)}{\pi_i(\tau)}\mathbf{p}_c(t)\mathbf{c}_i(\tau) = \pi_i(\tau)R_i(\tau)\mathbf{P}_{ci}(\tau)\mathbf{c}_i(\tau). \tag{12}$$

Hence, by $\pi_i(t) = \pi(t)$ and (12), *real* individual income, $\int_t^\infty (-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}_i(\tau)d\tau/\pi(t)$, is equal to the present value of real interest on future individual consumption, as stated in the following definition.

Definition 2 *Real individual income* at time t is determined as

$$Y_i(t) := \int_t^\infty \frac{\pi_i(\tau)}{\pi_i(t)} R_i(\tau)\mathbf{P}_{ci}(\tau)\mathbf{c}_i(\tau)d\tau.$$

By using (10) in (8), we can also express real individual income as the sum of current real individual consumption and the real individual savings. Furthermore, real individual incomes add exactly up to real national income since

$$\sum_{i\in I} Y_i(t) = \sum_{i\in I}\left(\frac{1}{\pi_i(t)}\int_t^\infty (-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}_i(\tau)d\tau\right)$$
$$= \frac{1}{\pi(t)}\int_t^\infty (-\dot{\mathbf{p}}_c(\tau))\sum_{i\in I}\mathbf{c}_i(\tau)d\tau$$
$$= \frac{1}{\pi(t)}\int_t^\infty (-\dot{\mathbf{p}}_c(\tau))\,\mathbf{c}(\tau)d\tau = Y(t),$$

where the first equality follows from (12), the second from $\pi_i(t) = \pi(t)$ for all $i \in I$, the third from (7), and the fourth from (6). We state these results as follows.

Proposition 2 Real individual income *at time t can be expressed as*

$$Y_i(t) = \mathbf{P}_c(t)\mathbf{c}_i(t) + \int_t^\infty \frac{\pi_i(\tau)}{\pi_i(t)}\mathbf{P}_c(\tau)\dot{\mathbf{c}}_i(\tau)d\tau.$$

Furthermore,

$$\sum_{i\in I} Y_i(t) = Y(t).$$

If all individuals consume goods in the same proportions, i.e., for each $i \in I$ there exists $\{\gamma_i(t)\}_{t=0}^\infty$ such that

$$\mathbf{c}_i(t) = \gamma_i(t)\mathbf{c}(t)$$

for all $t \geq 0$, then it follows from (3) and (9) that, for each $i \in I$, $\pi_i(\tau) = \pi(\tau)$ for all $\tau \geq t$. Hence, in this case the individual prices and interest rates as determined by (10) and (11) coincide with the real prices and real interest rate at a national level, as determined by (4) and (5).

6 Defining Sectoral Income

Divide the national economy into *n sectors*, so that each sector j is in $J := \{1,\ldots,n\}$. For each $j \in J$, denote by $\{\mathbf{x}^j(t)\}_{t=0}^\infty$ the path of sector j's vector of commodity flows excluding consumption flows acquired for end use, and let $\{\mathbf{p}_x(t)\}_{t=0}^\infty$ be the corresponding path of market (or calculated) *present value* prices of these commodity flows. Assume that aggregate cash flows are zero. Then the value of national consumption $\mathbf{c}$ equals the cash flow generated by $\sum_{j\in J}\mathbf{x}^j$ at each point in time:

$$\mathbf{p}_c(t)\mathbf{c}(t) = \sum_{j\in J}\mathbf{p}_x(t)\mathbf{x}^j(t)$$

for all $t \geq 0$. Furthermore, for each $i \in I$, let $\{\sigma_i^j(t)\}_{t=0}^\infty$ be the path of individual i's share of sector j, where for each $j \in J$,

$$\sum_{i\in I}\sigma_i^j(t) = 1 \quad \text{for all} \quad t \geq 0,$$

so that

$$\mathbf{p}_c(t)\mathbf{c}_i(t) = \sum_{j\in J} \sigma_i^j(t)\mathbf{p}_x(t)\mathbf{x}^j(t), \tag{13}$$

and consumer i's income from sector j equals $\int_t^\infty R_i(\tau)\sigma_i^j(\tau)\mathbf{p}_x(\tau)\mathbf{x}^j(\tau)d\tau$. By summing over individuals, we can now define each sector j's income by

$$\int_t^\infty \left(\sum_{i\in I} R_i(\tau)\sigma_i^j(\tau)\right)\mathbf{p}_x(\tau)\mathbf{x}^j(\tau)d\tau.$$

To define *real* sectoral income at time t in the same numeraire as real national income at time t, and obtain alternative expressions for sectoral income, define for each $j \in J$ the path of market (or calculated) real consumption interest rates $\{R^j(\tau)\}_{\tau=t}^\infty$ for sector j as a weighted average of the individual interest rates,

$$R^j(\tau) := \sum_{i\in I} \sigma_i^j(\tau)R_i(\tau)$$

for all $\tau \geq t$, and derive a Divisia consumer price index for sector j from its path of real consumption interest rates,

$$\pi^j(\tau) := \pi(t)e^{-\int_t^\tau R^j(s)ds}$$

for all $\tau \geq t$. Clearly we have that $\pi^j(t) = \pi(t)$ and

$$R^j(\tau) = -\dot{\pi}^j(\tau)/\pi^j(\tau) \tag{14}$$

for all $\tau \geq t$. Define the path of market (or calculated) *real* commodity prices $\{\mathbf{P}_x^j(\tau)\}_{\tau=t}^\infty$ for sector j by

$$\mathbf{P}_x^j(\tau) = \mathbf{p}_x(\tau)/\pi^j(\tau)$$

for all $\tau \geq t$. By these definitions we obtain

$$\left(\sum_{i\in I} R_i(\tau)\sigma_i^j(\tau)\right)\mathbf{p}_x(\tau)\mathbf{x}^j(\tau)=\pi^j(\tau)R^j(\tau)\mathbf{P}_x^j(\tau)\mathbf{x}^j(\tau)=\left(-\dot{\pi}^j(\tau)\right)\mathbf{P}_x^j(\tau)\mathbf{x}^j(\tau). \tag{15}$$

Hence, by $\pi^j(t) = \pi(t)$ and (15), *real* sectoral income, $\int_t^\infty \left(\sum_{i\in I} R_i(\tau)\sigma_i^j(\tau)\right)\mathbf{p}_x(\tau)$ $\mathbf{x}^j(\tau)d\tau/\pi(t)$, is equal to the present value of real interest on future sectoral cash flow, as stated in the following definition.

Definition 3 *Real sectoral income* at time t is determined as

$$Y^j(t) := \int_t^\infty \frac{\pi^j(\tau)}{\pi^j(t)}R^j(\tau)\mathbf{P}_x^j(\tau)\mathbf{x}^j(\tau)d\tau.$$

Differentiation of $\pi^j(\tau)\mathbf{P}_x^j(\tau)\mathbf{x}(\tau)$ yields

$$\frac{d}{dt}\left(\pi^j(\tau)\mathbf{P}_x^j(\tau)\mathbf{x}(\tau)\right) = \dot{\pi}^j(\tau)\mathbf{P}_x^j(\tau)\mathbf{x}(\tau) + \pi^j(\tau)\dot{\mathbf{P}}_x^j(\tau)\mathbf{x}(\tau) + \pi^j(\tau)\mathbf{P}_x^j(\tau)\dot{\mathbf{x}}(\tau).$$

Integrating on both sides under the assumption that $\pi^j(\tau)\mathbf{P}_x^j(\tau)\mathbf{x}(\tau) \to 0$ as $t \to \infty$, leads to the following equation:

$$-\pi^j(t)\mathbf{P}_x^j(t)\mathbf{x}(t) = \int_t^\infty \dot{\pi}^j(\tau)\mathbf{P}_x^j(\tau)\mathbf{x}(\tau)d\tau + \int_t^\infty \pi^j(\tau)\dot{\mathbf{P}}_x^j(\tau)\mathbf{x}(\tau)d\tau + \int_t^\infty \pi^j(\tau)\mathbf{P}_x^j(\tau)\dot{\mathbf{x}}(\tau)d\tau.$$

By rearranging this equality, invoking (14) and applying Definition 3, we obtain the first part of Proposition 3 below. The second part of the proposition follows since

$$\begin{aligned}
\sum_{j\in J}\left(\sum_{i\in I} R_i(\tau)\sigma_i^j(\tau)\right)\mathbf{p}_x(\tau)\mathbf{x}^j(\tau) &= \sum_{i\in I} R_i(\tau)\left(\sum_{j\in J}\sigma_i^j(\tau)\mathbf{p}_x(t)\mathbf{x}^j(\tau)\right)\\
&= \sum_{i\in I}\left(-\frac{\dot{\pi}_i(\tau)}{\pi_i(\tau)}\right)\mathbf{p}_c(\tau)\mathbf{c}_i(\tau)\\
&= \sum_{i\in I}\left(-\dot{\mathbf{p}}_c(\tau)\right)\mathbf{c}_i(\tau) = \left(-\dot{\mathbf{p}}_c(\tau)\right)\mathbf{c}(\tau),
\end{aligned}$$

using (9), (11) and (13); i.e., real sectoral incomes add exactly up to real national income:

$$\begin{aligned}
\sum_{j\in J} Y^j(0) &= \sum_{j\in J}\left(\frac{1}{\pi(t)}\int_t^\infty\left(\sum_{i\in I} R_i(\tau)\sigma_i^j(\tau)\mathbf{p}_x(\tau)\mathbf{x}^j(\tau)d\tau\right)\right)\\
&= \frac{1}{\pi(t)}\int_t^\infty\left(-\dot{\mathbf{p}}_c(\tau)\right)\mathbf{c}(\tau)d\tau = Y(0).
\end{aligned}$$

Proposition 3 Real sectoral income *at time t can be expressed as*

$$Y^j(t) = \underbrace{\mathbf{P}_x^j(t)\mathbf{x}(t)}_{\text{current cash flow}} + \underbrace{\int_t^\infty \frac{\pi^j(\tau)}{\pi^j(t)}\mathbf{P}_x^j(\tau)\dot{\mathbf{x}}(\tau)d\tau}_{\text{sectoral net investments}} + \underbrace{\int_t^\infty \frac{\pi^j(\tau)}{\pi^j(t)}\dot{\mathbf{P}}_x^j(\tau)\mathbf{x}(\tau)d\tau}_{\text{price change effects}}.$$

Furthermore,

$$\sum_{j\in J} Y^j(t) = Y(t).$$

The first part of Proposition 3 means that we are able to split the sector's income into its current cash flow, its net investments, and its price change effects, by using a consumer price index associated with the use of the cash flow from sector j.

For each sector j, $R^j(\tau) = \sum_{i\in I} R_i(\tau)\sigma_i^j(\tau)$ is an average real consumption interest rate to be used for the calculation of sector j's income. If the individual interest rates as determined by (11) coincide with the real interest rate at a national level as determined by (5), then the average consumption interest for each sector j also coincide with the real interest at a national level. Also if, for each $i \in I$, $\sigma_i^j(\tau) = \sigma_i(\tau)$ for all $j \in N$, then it follows that the average consumption interest for each sector j coincide with the real interest rate at a national level.

In this section, "sector" has been used as an abstract term. The examples of the two next sections illustrate ways in which an economy can be divided into different "sectors".

7 Functional Income Shares in a DHS Model

Consider the Cobb-Douglas Dasgupta-Heal-Solow (DHS) model (Dasgupta and Heal 1974, 1979; Solow 1974). Hence, production, $q(t)$ at time t is given by

$$q(t) = k(t)^{\alpha} r(t)^{\beta}$$

where k is the capital stock, r is resource input being extracted at no cost from a finite stock, and the available labor ℓ is constant and normalized to one (i.e, $\ell(t) = 1$ for all t), and where we assume that

$$1 > \alpha + \beta > \alpha > \beta.$$

Production can be split into consumption $c(t)$ and accumulation of capital $\dot{k}(t)$:

$$q(t) = c(t) + \dot{k}(t).$$

Since this is a one-consumption good model, price indices need not be invoked. Consequently, the real price of consumption can be set to 1 for all $t \geq 0$, and the real wage $P_{\ell}(t)$, the real price of resource input $P_r(t)$, and the real interest rate $R(t)$ equals the marginal productivities of inputs:

$$\begin{aligned} P_{\ell}(t) &= (1 - \alpha - \beta) q(t), \\ P_r(t) &= \beta q(t)/r(t), \\ R(t) &= \alpha q(t)/k(t). \end{aligned} \tag{16}$$

Furthermore, along an efficient path, the Hotelling rule,

$$\pi(t) P_r(t) = P_r(0),$$

is satisfied, where $\{\pi(t)\}_{t=0}^{\infty}$ is the path of present value prices of consumption:

$$\pi(t) = e^{-\int_0^t R(\tau) d\tau} \tag{17}$$

for all $t \geq 0$.

Assume that the economy follows the efficient constant consumption path, which exists under these assumption. This path is characterized by a constant production q, with the constant consumption being a fixed share of production: $c = (1 - \beta) q$, with the reminder being used for capital accumulation:

$$\dot{k}(t) = \beta q. \tag{18}$$

Consider three sectors, corresponding to the supply of labor, the supply of resource input, and the production sector. We assume that the production sector owns the capital stock and is responsible for capital accumulation. The cash flow to each of these sectors at each point in time is as follows:

$$\begin{array}{ll} \text{Labor:} & P_\ell(t) = (1-\alpha-\beta)q \\ \text{Resource:} & P_r(t)r(t) = \beta q \\ \text{Production/Capital:} & R(t)k(t) - \dot{k}(t) = (\alpha-\beta)q \end{array} \tag{19}$$

It is easy to check that the cash flow from these sectors add up to national consumption at each point in time. In order to find sectoral income, note that

$$\int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)d\tau = \frac{1}{\pi(t)} \int_t^\infty (-\dot{\pi}(\tau))\, d\tau = \frac{\pi(t)}{\pi(t)} = 1, \tag{20}$$

provided that $\pi(\tau) \to 0$ as $\tau \to \infty$. This implies that sectoral income at time t is given by

$$\begin{array}{ll} \text{Labor:} & Y^\ell(t) = \displaystyle\int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)(1-\alpha-\beta)q d\tau = (1-\alpha-\beta)q \\ \text{Resource:} & Y^r(t) = \displaystyle\int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)\beta q d\tau = \beta q \\ \text{Production/Capital:} & Y^k(t) = \displaystyle\int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)(\alpha-\beta)q d\tau = (\alpha-\beta)q \end{array}$$

Since the real interest rate is decreasing over time, resource income is lower than the interest on the resource wealth and production/capital income is lower than the interest on capital, as demonstrated in Appendix B.

Alternatively, we can use Proposition 3 to derive expressions for sectoral income.

$$\begin{aligned} Y^\ell(t) &= P_\ell(t)\ell(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_\ell(\tau)\dot{\ell}(\tau)d\tau + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{P}_\ell(\tau)\ell(\tau)d\tau \\ &= P_\ell(t) = (1-\alpha-\beta)q \quad \text{since } P_\ell(\tau) = (1-\alpha-\beta)q \text{ and } \ell(\tau) = 1 \text{ for all } \tau. \end{aligned}$$

Hence, for labor, there is no net investments or price change effects.

$$\begin{aligned} Y^r(t) &= P_r(t)r(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} P_r(\tau)\dot{r}(\tau)d\tau + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{P}_r(\tau)r(\tau)d\tau \\ &= P_r(t)r(t) + P_r(t)\int_t^\infty \dot{r}(\tau)d\tau + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{P}_r(\tau)r(\tau)d\tau \quad \text{by Hotelling's rule,} \\ &= \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{P}_r(\tau)r(\tau)d\tau \qquad \text{since } \lim_{\tau\to\infty} r(\tau) = 0 \text{ implies } \int_t^\infty \dot{r}(\tau)d\tau = -r(t), \\ &= \frac{1}{\pi(t)} \int_t^\infty (-\dot{\pi}(\tau))\, P_r(\tau)r(\tau)d\tau \qquad \text{since } \dot{\pi} P_r + \pi \dot{P}_r = 0 \text{ by Hotelling's rule,} \\ &= \beta q \qquad \text{by (20) since } P_r(\tau)r(\tau) = \beta q \text{ for all } \tau. \end{aligned}$$

This means that resource income can be split like this:

$$Y^r(t) := \underbrace{\beta q}_{\text{current cash flow}} - \underbrace{\beta q}_{\text{net investments}} + \underbrace{\beta q}_{\text{price change effects}},$$

where the negative net investments equal the Hotelling rents and cancel out the value of production.

$$\begin{aligned}
Y^k(t) &= R(t)k(t) - \dot{k}(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \left(R(\tau)\dot{k}(\tau) - \ddot{k}(\tau)\right) d\tau + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{R}(\tau)k(\tau)d\tau \\
&= R(t)k(t) - \dot{k}(t) + \dot{k}(t) + \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \dot{R}(\tau)k(\tau)d\tau \quad \text{by (20) as } \dot{k}(\tau) = \beta q \text{ for all } \tau, \\
&= R(t)k(t) - \int_t^\infty \frac{\pi(\tau)}{\pi(t)} R(\tau)\dot{k}(\tau)d\tau \quad \text{since } \dot{R}(\tau)/R(\tau) = -\dot{k}(\tau)/k(\tau) \text{ for all } \tau, \\
&= R(t)k(t) - \dot{k}(t) = (\alpha - \beta)q \qquad \text{by (20) as } \dot{k}(\tau) = \beta q \text{ for all } \tau,
\end{aligned}$$

This means that production/capital income can be split like this:

$$Y^k(t) := \underbrace{(\alpha - \beta)q}_{\text{current cash flow}} + \underbrace{\beta q}_{\text{net investments}} - \underbrace{\beta q}_{\text{price change effects}},$$

so that the negative price change effects due to the decreasing interest rate cancel out the positive net investments.

8 Income of a Reservoir

Consider a reservoir of a resource, say petroleum. The reservoir has a fixed size $S(0)$, and non-negative resource extraction r depends on non-negative extractive effort e through an increasing, strictly concave, and continuously differentiable production function F,

$$r = F(e),$$

satisfying $F(0) = 0$. The real prices of the extracted resource and extractive effort are constant in real terms and equal P_r and P_e respectively. Hence the real cash-flow at time t is given by

$$P_r r(t) - P_e e(t),$$

where $r(t) = F(e(t))$ for all $t \geq 0$. By defining the cost function C by

$$C(r) = P_e F^{-1}(r) \tag{21}$$

for all r in the range of F, the real cash-flow at time t can be rewritten as follows:

$$P_r r(t) - C(r(t)) = \left(P_r - C'(r(t))\right) r(t) + \left(C'(r(t))r(t) - C(r(t))\right).$$

Under the assumption that there is a constant interest real rate R, so that $\pi(t) = e^{-Rt}$, and the reservoir is extracted in a profit-maximizing manner, we have that

$$e^{-Rt}\left(P_r - C'(r(t))\right) = \text{constant} \tag{22}$$

for all $t \in [0, T]$, where T is the time at which the reservoir is exhausted:

$$\int_0^T r(t)dt = S(0).$$

Hence, by (22) the income at time t of the reservoir can be written as

$$\begin{aligned} Y^r(t) &= \int_t^T e^{-R(\tau-t)} R\left(P_r r(\tau) - C(r(\tau))\right) d\tau \\ &= R\left(P_r - C'(r(t))\right) S(t) + R \int_t^T e^{-R(\tau-t)} \left(C'(r(\tau)) r(\tau) - C(r(\tau))\right) d\tau. \end{aligned}$$

The first term is interest on the present value of future Hotelling rent, while the second term is interest on the present value of future Ricardian rent.

Alternatively, we can use Proposition 3 to derive expressions for the income of a reservoir. Since P_r and P_e are assumed to be constant, we obtain

$$\begin{aligned} Y^r(t) &= P_r(t)r(t) - P_e(t)e(t) + \int_t^\infty e^{-R(\tau-t)} \left(P_r(\tau)\dot{r}(\tau) - P_e(\tau)\dot{e}(\tau)\right) d\tau \\ &= P_r(t)r(t) - C(r(t)) + \int_t^\infty e^{-R(\tau-t)} \left(P_r(\tau) - C'(r(\tau))\right) \dot{r}(\tau) d\tau \qquad \text{by (21)}, \\ &= P_r(t)r(t) - C(r(t)) + \left(P_r(t) - C'(r(t))\right) \int_t^\infty \dot{r}(\tau) d\tau \qquad \text{by (22)}, \\ &= C'(r(t))r(t) - C(r(t)) \qquad \text{since } \lim_{\tau\to\infty} r(\tau) = 0 \text{ implies } \int_t^\infty \dot{r}(\tau) d\tau = -r(t). \end{aligned}$$

Hence, we arrive at the result that the income of a reservoir—given the assumptions that we have made—equals current Ricardian rent.

By interpreting F to be derived from a constant returns to scale production function $\tilde{F}$ that depends on both effort e and the ground g from which the resource is extracted, by setting $g = 1$, i.e.,

$$F(e) = \tilde{F}(e, 1),$$

we obtain the interpretation that the income of a reservoir is equal to the marginal productivity of the ground evaluated at the resource price net of the Hotelling rent:

$$C'(r) \frac{\partial \tilde{F}(e, 1)}{\partial g} = C'(r)\left(F(e) - F'(e)e\right) = C'(r)r - C(r),$$

since $C'(r)F'(e) = P_e$ and $P_e e = C(r)$ by the definition of C.

9 Concluding Remarks

In this paper we have presented a definition of sectoral income which is compatible with an important line of welfare-based theory of comprehensive national accounting in the tradition of Hicks (1946), Samuelson (1961), Weitzman (1976) and Sefton and Weale (2006). The definition has the desirable properties that sectoral income summed over all sectors adds up to a concept of national income which

- is a guide to prudent behavior in the sense that dynamic welfare improves if and only if consumption is less than national income, and
- equals net national product in a closed economy with a stationary technology.

We have decomposed sectoral income into current cash flow, sectoral net investments and price change effects. Our definition of sectoral income and its decomposition have been illustrated though application to a general model of capital resource accumulation and resource depletion as well as a partial model of a single resource reservoir.

As noted in Sect. 2, we do not require that a discounted utilitarian welfare function is maximized. Rather, as explained in Appendix A, the formal analysis builds on the assumption that the economy's actual decisions are taken according to a *resource allocation mechanism* (as introduced by Dasgupta and Mäler 2000; Dasgupta 2001; Arrow et al. 2003). The resource allocation mechanism is allowed to be inefficient, due to, e.g., externalities, monopolistic competition, or distortionary taxation.[3] This is relevant in a world facing serious environmental problems caused by uninternalized externalities. In particular, how is the income of an economy's petroleum sector affected by the fact that both petroleum extraction and petroleum use cause greenhouse gas emissions? It is not trivial to calculate the relevant accounting prices under such conditions. Guidelines for practical calculation of accounting prices are outside the scope of the present paper. Arrow et al. (2003) discuss problems that arise within such a framework and is a useful reference.

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Appendix A: Welfare Results[4]

If *dynamic welfare* is forward-looking and numerically representable, then dynamic welfare, denoted V, is a functional $\mathcal{G}$ of the path of consumption flows:

$$V(t) = \mathcal{G}(\{\mathbf{c}(\tau)\}_{\tau=t}^{\infty}, t). \tag{A1}$$

The functional $\mathcal{G}$ is *time-invariant* if

$$\mathcal{G}(\{\mathbf{c}(\tau)\}_{\tau=t}^{\infty}, t) = \mathcal{G}(\{\tilde{\mathbf{c}}(\tau)\}_{\tau=0}^{\infty}, 0). \tag{A2}$$

where $\{\tilde{\mathbf{c}}(\tau)\}_{\tau=0}^{\infty}$ is determined by $\tilde{\mathbf{c}}(\tau) = \mathbf{c}(t+\tau)$ for all $\tau \geq 0$. Furthermore, $\mathcal{G}$ satisfies a condition of *independent future* if

$$\mathcal{G}(\{\mathbf{c}'(\tau)\}_{\tau=0}^{\infty}, 0) < \mathcal{G}(\{\mathbf{c}''(\tau)\}_{\tau=0}^{\infty}, 0) \quad \Leftrightarrow \quad \mathcal{G}(\{\mathbf{c}'(\tau)\}_{\tau=t}^{\infty}, t) < \mathcal{G}(\{\mathbf{c}''(\tau)\}_{\tau=t}^{\infty}, t)$$

whenever $\{\mathbf{c}'(\tau)\}_{\tau=0}^{\infty}$ and $\{\mathbf{c}''(\tau)\}_{\tau=0}^{\infty}$ coincides during the interval $[0, t]$.

[3] The general results of Appendix A (Propositions 4 and 5) that we use to justify the national income definition do not require that dynamic welfare is maximized. However, the result of Appendix A (Proposition 6) that we use to justify the individual income definition requires that individuals make decisions according to the relevant accounting prices. The propositions of the main text remain valid even if the premise of Proposition 6 does not hold.

[4] This appendix is partly based on analysis and results in Asheim (2007).

Proposition 4 *Let dynamic welfare be numerically representable by a forward-looking and time-invariant functional $\mathcal{G}$ of the path of consumption flows, satisfying a condition of independent future. Then, if $\mathcal{G}$ is smooth, there exists a path of present value consumer prices $\{\mathbf{p}_c(t)\}_{t=0}^{\infty}$ such that welfare improves along the implemented path $\{\mathbf{c}(t)\}_{t=0}^{\infty}$ at time t if and only if*

$$\int_t^{\infty} \mathbf{p}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau \ > \ 0.$$

Proof Since $\mathcal{G}$ is smooth and satisfies independent future, there exists a path of present value consumer prices $\{\mathbf{p}_c(t)\}_{t=0}^{\infty}$, unique up to a choice of numeraire, supporting the implemented path of consumption flows $\{\mathbf{c}(t)\}_{t=0}^{\infty}$ in the sense that, for all t,

$$\lambda(t)dV(t) = \int_t^{\infty} \mathbf{p}_c(\tau)d\mathbf{c}(\tau). \tag{A3}$$

for some $\lambda(t) > 0$. Furthermore, since $\mathcal{G}$ is time-invariant,

$$\begin{aligned}\lambda(t)\dot{V}(t) = \lambda(t)\frac{d}{dt}(\mathcal{G}(\{\mathbf{c}(\tau)\}_{\tau=t}^{\infty}, t)) &= \lambda(t)\frac{d}{dt}(\mathcal{G}(\{\mathbf{c}(t+\tau)\}_{\tau=0}^{\infty}, 0)) \\ &= \int_t^{\infty} \mathbf{p}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau, \end{aligned} \tag{A4}$$

where the first equality follows from (A1), the second from (A2), and the third from (A3), thus establishing the result that local welfare comparisons across time for a given path of consumption flows $\{\mathbf{c}(t)\}_{t=0}^{\infty}$ depends on the present value of future consumption growth. □

Example Discounted utilitarianism. Let the welfare functional be given as

$$\mathcal{G}(\{\mathbf{c}(\tau)\}_{\tau=t}^{\infty}, t) = \int_t^{\infty} e^{-\rho(\tau-t)}u(\mathbf{c}(\tau))d\tau.$$

Then it follows that

$$\begin{aligned}\frac{d}{dt}\left(\int_t^{\infty} e^{-\rho(\tau-t)}u(\mathbf{c}(\tau))d\tau\right) &= -u(\mathbf{c}(t)) + \rho\int_t^{\infty} e^{-\rho(\tau-t)}u(\mathbf{c}(\tau))d\tau \\ &= e^{\rho t}\int_t^{\infty} e^{-\rho(\tau-t)}\nabla u(\mathbf{c}(\tau))\dot{\mathbf{c}}(\tau)d\tau,\end{aligned} \tag{A5}$$

where the second equality follows by integrating by parts. This verifies (A4) in the case of discounted utilitarianism by setting, for all t, $\lambda(t) = e^{-\rho t}$ and $\mathbf{p}_c(t) = e^{-\rho t}\nabla u(\mathbf{c}(t))$. Note also that time-invariance corresponds to

$$\int_t^{\infty} e^{-\rho(\tau-t)}u(\mathbf{c}(\tau))d\tau = \int_0^{\infty} e^{-\rho(\tau-0)}u(\mathbf{c}(t+\tau))d\tau,$$

which clearly holds in the case of discounted utilitarianism.

Consider that the economy's actual decisions are taken according to a *resource allocation mechanism* (RAM) that assigns some attainable consumption-net investment pair $(\mathbf{c}, \dot{\mathbf{k}})$ to any vector of capital stocks $\mathbf{k}$. Hence, for any vector of capital stocks $\mathbf{k}$, the RAM determines the consumption and net investment flows. The net investment flows in turn map out the development of the capital stocks. The RAM thereby implements a feasible path of consumption flows, net investment flows, and capital stocks, for any initial vector of capital stocks. If the set of all attainable consumption-net investment pairs to any vector of capital stocks $\mathbf{k}$ is time-invariant (i.e., in closed economy with a stationary technology), one can assume that the RAM in the economy is *Markovian* (i.e., the chosen consumption-net investment pair depends only on $\mathbf{k}$) and time-invariant. If the welfare functional $\mathcal{G}$ is time-invariant and a stationary (i.e., Markovian and time-invariant) RAM implements a unique path, then the dynamic welfare at t of the implemented path,

$$V(t) = \mathcal{V}(\mathbf{k}(\mathbf{t})),$$

depends solely on the current vector of capital stocks $\mathbf{k}(t)$.

Proposition 5 *Let* (1) *dynamic welfare be numerically representable by a forward-looking and time-invariant functional $\mathcal{G}$ of the path of consumption flows, satisfying a condition of independent future, and* (2) *the* RAM *be stationary. Then, if $\mathcal{G}$ is smooth, the* RAM *implements a unique path, and the state valuation function $\mathcal{V}$ is differentiable, there exist paths of present value consumer prices $\{\mathbf{p}_c(t)\}_{t=0}^{\infty}$ and present value net investment prices $\{\mathbf{p}_k(t)\}_{t=0}^{\infty}$ such that welfare improves along the implemented path $\{\mathbf{c}(t), \dot{\mathbf{k}}(t), \mathbf{k}(t)\}_{t=0}^{\infty}$ at time t if and only if*

$$\int_t^{\infty} \mathbf{p}_c(\tau)\dot{\mathbf{c}}(\tau)d\tau = \mathbf{p}_k(t)\dot{\mathbf{k}}(t) > 0.$$

Proof By the proof of Proposition 4, there exists a path of supporting present value consumer prices $\{\mathbf{p}_c(t)\}_{t=0}^{\infty}$, unique up to a choice of numeraire, such that, for all t, (A3) is satisfied for some $\lambda(t) > 0$. Since $\mathcal{V}$ exists and is differentiable, there exists, at any t, a supporting vector of capital prices $\mathbf{p}_k(t)$ satisfying

$$\lambda(t)\nabla\mathcal{V}(\mathbf{k}(t)) = \mathbf{p}_k(t).$$

Hence, since $\mathcal{V}$ is time-invariant, local welfare comparisons across time for a given implemented path $\{\mathbf{c}(t), \dot{\mathbf{k}}(t), \mathbf{k}(t)\}_{t=0}^{\infty}$ depends on the value of net investments:

$$\lambda(t)\dot{V}(t) = \lambda(t)\nabla\mathcal{V}(\mathbf{k}(t))\dot{\mathbf{k}}(t) = \mathbf{p}_k(t)\dot{\mathbf{k}}(t). \tag{A6}$$

The result follows by combining (A4) and (A6). □

Example Discounted utilitarianism (continued). Equation (A5) can be rewritten as

$$\nabla\mathcal{V}(\mathbf{k}(t))\dot{\mathbf{k}}(t) = -u(\mathbf{c}(t)) + \rho\mathcal{V}(\mathbf{k}(t))$$

or

$$u(\mathbf{c}(t)) + \nabla\mathcal{V}(\mathbf{k}(t))\dot{\mathbf{k}}(t) = \rho\mathcal{V}(\mathbf{k}(t)).$$

Differentiating once more w.r.t. time yields:

$$\nabla u(\mathbf{c}(t))\dot{\mathbf{c}}(t) + \frac{d[\nabla\mathcal{V}(\mathbf{k}(t))\dot{\mathbf{k}}(t)]}{dt} = \rho\nabla\mathcal{V}(\mathbf{k}(t))\dot{\mathbf{k}}(t),$$

or equivalently, since $\mathbf{p}_c(t) = e^{-\rho t}\nabla u(\mathbf{c}(t))$ and $\mathbf{p}_k(t) = e^{-\rho t}\nabla\mathcal{V}(\mathbf{k}(t))$,

$$\mathbf{p}_c(t)\dot{\mathbf{c}}(t) = -\frac{d(\mathbf{p}_k(t)\dot{\mathbf{k}}(t))}{dt}$$

as $d\left(\nabla\mathcal{V}(\mathbf{k}(t))\dot{\mathbf{k}}\right)/dt = d\left(e^{\rho t}\mathbf{p}_k\dot{\mathbf{k}}\right)/dt = e^{\rho t}\left(d(\mathbf{p}_k\dot{\mathbf{k}})/dt + \rho\mathbf{p}_k\dot{\mathbf{k}}\right)$. This means that the equality in Proposition 5 follows through integration, provided that the following net investment value transversality condition holds:

$$\lim_{t\to\infty}\mathbf{p}_c(t)\dot{\mathbf{k}}(t) = 0.$$

If *individual welfare* is forward-looking and numerically representable, then individual welfare, denoted V_i, is a functional $\mathcal{G}_i$ of the path of individual consumption flows:

$$V_i(t) = \mathcal{G}_i(\{\mathbf{c}_i(\tau)\}_{\tau=t}^{\infty}, t).$$

Proposition 6 *Let individual welfare be numerically representable by a forward-looking and time-invariant functional $\mathcal{G}_i$ of the path of individual consumption flows, satisfying a condition of independent future. Then, if $\mathcal{G}_i$ is smooth and the individual path $\{\mathbf{c}_i(t)\}_{t=0}^{\infty}$ maximizes individual welfare subject to the budget constraint $\int_0^\infty \mathbf{p}_c(t)\mathbf{c}_i(t)dt = m(0)$, individual welfare improves along the implemented path at time t if and only if*

$$\int_t^\infty \mathbf{p}_c(\tau)\dot{\mathbf{c}}_i(\tau)d\tau > 0.$$

Proof Since $\mathcal{G}_i$ is smooth and satisfies independent future, and $\{\mathbf{c}_i(t)\}_{t=0}^{\infty}$ maximizes individual welfare subject to the budget constraint $\int_0^\infty \mathbf{p}_c(t)\mathbf{c}_i(t)dt = m(0)$, it follows that, for all t,

$$\lambda_i(t)dV_i(t) = \int_t^\infty \mathbf{p}_c(\tau)d\mathbf{c}_i(\tau).$$

for some $\lambda_i(t) > 0$. The result follows by applying the argument of (A4) to $\mathcal{G}_i$. □

Appendix B: Other Income Concepts

In this appendix we define formally the concepts of *income as interest on wealth* and *wealth equivalent income*, as alternative concepts to the one analyzed in the main part of this paper. Both these wealth-based concepts of income depend on the way in which nominal prices are turned into real prices, thus raising an indexing issue. To sidestep this issue we choose to define these alternative income concepts in a setting where consumption is a scalar.

With only one consumption good, the basic setting of the main text can be restated as follows: Let $\{c(t)\}_{t=0}^{\infty}$ be the path of consumption flows in the economy, and let $\{\pi(t)\}_{t=0}^{\infty}$ be the corresponding path of positive market (or calculated) *present value* prices of consumption, with $\int_0^\infty \pi(\tau)d\tau < \infty$. Define the path of market (or calculated) *real* consumption interest rates $\{R(t)\}_{t=0}^{\infty}$ by $R(t) = -\dot{\pi}(t)/\pi(t)$ for all $t \geq 0$. For each $j \in J := \{1, \ldots, n\}$, denote by $\{\mathbf{x}^j(t)\}_{t=0}^{\infty}$ the path of sector j's commodity flows excluding consumption flow acquired for end use, and let $\{\mathbf{p}_x(t)\}_{t=0}^{\infty}$ be the corresponding path of market (or calculated) *present value* prices of these commodity flows. Assume that aggregate cash holdings are zero. Then the value of national consumption equals total cash flow at each point in time: $\pi(t)c(t) = \sum_{j\in J} \mathbf{p}_x(t)\mathbf{x}^j(t)$ for all $t \geq 0$. In this setting, real income as the present value of real interest on future consumption (at the aggregate level) and cash flow (at the sectoral level) at time t is determined as follows:

$$\text{At the national level:} \qquad Y(t) := \int_t^{\infty} R(\tau)\frac{\pi(\tau)}{\pi(t)}c(\tau)d\tau$$

$$\text{At the sectoral level:} \qquad Y^j(t) := \int_t^{\infty} R(\tau)\frac{\mathbf{p}_x^j(\tau)}{\pi(t)}\mathbf{x}^j(\tau)d\tau.$$

For the wealth-based income concepts below, we must define real wealth at time t:

$$\text{At the national level:} \qquad W(t) := \int_t^{\infty} \frac{\pi(\tau)}{\pi(t)}c(\tau)d\tau$$

$$\text{At the sectoral level:} \qquad W^j(t) := \int_t^{\infty} \frac{\mathbf{p}_x^j(\tau)}{\pi(t)}\mathbf{x}^j(\tau)d\tau.$$

Income as Interest on Wealth

Definition 4 *Real income as interest on wealth* at time t is determined as follows:

$$\text{At the national level:} \qquad Z(t) := R(t)W(t) = c(t) + \dot{W}(t)$$

$$\text{At the sectoral level:} \qquad Z^j(t) := R(t)W^j(t) = \frac{\mathbf{p}_x(t)}{\pi(t)}\mathbf{x}^j(t) + \dot{W}^j(t).$$

Since the current value of sectoral cash flow sum up to national consumption for all $t \geq 0$, it follows that real income as interest on wealth has the property that sectoral income summed over all sectors add up to national income, $Z(t) = \sum_{j\in J} Z^j(t)$.

In view of Proposition 5 of Appendix A we can check whether real income as interest on wealth at the national level equals net national product in a closed economy with a stationary technology by comparing $Z(t)$ with $Y(t)$. By Sefton and Weale (2006, Eq. 19),

$$Y(t) - Z(t) = \int_t^{\infty} \dot{R}(\tau)\frac{\pi(\tau)}{\pi(t)}W(\tau)d\tau.$$

Hence, the definition of real income as interest on wealth is unproblematic as long as the economy is in a steady state with a constant real interest rate. In this case, $Z(t)$ can also serve as indicator for prudent behavior, since we obtain

$$Z(t) - c(t) = \dot{W}(t) = \int_t^{\infty} \frac{\pi(\tau)}{\pi(t)}\dot{c}(\tau)d\tau + Z(t) - Y(t)$$

by writing $W(t) = \int_0^{\infty} (\pi(t+\tau)/\pi(t))\, c(t+\tau)d\tau$ and differentiating with respect to t.

However, there are important models where there exists no steady state and where the real interest rate is not constant. In particular, in the DHS model—which represents a closed economy with a stationary technology and which we have presented in its Cobb-Douglas version in Sect. 7—the real interest rate is decreasing along a path where capital is accumulated and resource flow diminishes. Indeed, (17) and (18) implies that the real interest is given by

$$R(t) = \frac{\alpha}{k(0)/q + \beta t} \tag{A7}$$

along an efficient path with constant consumption in the DHS model, where α and β are the functional shares of capital and resource input, $k(0)$ is the initial capital stock, and q is the constant production along the egalitarian path. Since (17) and (A7) imply that

$$R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} d\tau = \frac{\alpha}{\alpha - \beta},$$

for all $t \geq 0$, it follows from (19) that sectoral income as defined in Definition 4 is given by

Labor: $$Z^\ell(t) = R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} (1 - \alpha - \beta) q d\tau = \frac{\alpha}{\alpha - \beta} (1 - \alpha - \beta) q$$

Resource: $$Z^r(t) = R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} \beta q d\tau = \frac{\alpha}{\alpha - \beta} \beta q$$

Production/Capital: $$Z^k(t) = R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} (\alpha - \beta) q d\tau = \frac{\alpha}{\alpha - \beta} (\alpha - \beta) q.$$

Moreover, since consumption equals $(1 - \beta)q$, income at the national level is given by

$$Z(t) = R(t) \int_t^\infty \frac{\pi(\tau)}{\pi(t)} (1 - \beta) q d\tau = \frac{\alpha}{\alpha - \beta} (1 - \beta) q.$$

Since net national product equals $(1 - \beta)q$ along the egalitarian path, it now follows that national income as defined in Definition 4 will exceed net national product and exceed consumption along an efficient path with constant consumption. This contradicts that national income as interest on wealth equals net national product and also means that real income as interest on wealth can not serve as an indicator for prudent behavior.

Wealth Equivalent Income

Real wealth equivalent income is the stream, constant in real terms, yielding the same wealth as the implemented path. Writing H for such income, with reference to Hicks (1946), this leads to the following defining equations.

At the national level: $$\int_t^\infty \pi(\tau) H(t) d\tau = \int_t^\infty \pi(\tau) c(\tau) d\tau$$

At the sectoral level: $$\int_t^\infty \pi(\tau) H^j(t) d\tau = \int_t^\infty \mathbf{p}_x(\tau) \mathbf{x}^j(\tau) d\tau.$$

Solving with respect to wealth equivalent income and defining the infinitely long-term interest rate, R_∞ by

$$R_\infty(t) := \frac{\pi(t)}{\int_t^\infty \pi(\tau)d\tau} = \frac{\int_t^\infty \pi(\tau)R(\tau)d\tau}{\int_t^\infty \pi(\tau)d\tau}. \tag{A8}$$

we obtain the expressions of Definition 5.

Definition 5 *Real wealth equivalent income* at time t is determined as follows:

$$\text{At the national level:} \qquad H(t) := \frac{\int_t^\infty \pi(\tau)c(\tau)d\tau}{\int_t^\infty \pi(\tau)d\tau} = R_\infty(t)W(t)$$

$$\text{At the sectoral level:} \qquad H^j(t) := \frac{\int_t^\infty \mathbf{p}_x(\tau)\mathbf{x}^j(\tau)d\tau}{\int_t^\infty \pi(\tau)d\tau} = R_\infty(t)W^j(t).$$

Note that (A8) entails that $R_\infty(t)$ is a discount-weighted average of $\{R(\tau)\}_{t=\tau}^\infty$; hence, $R_\infty(t') < R_\infty(t) < R(t)$ for $t' > t$ if $R(\tau)$ is a decreasing function of τ for $\tau \geq t$.

Since the current value of sectoral cash flow sum up to national consumption for all $t \geq 0$, it follows that real wealth equivalent income has the property that sectoral income summed over all sectors add up to national income, $H(t) = \sum_{j\in J} H^j(t)$. Moreover, it also follows that

$$\dot{H}(t) = R_\infty(t)\left(H(t) - c(t)\right),$$

entailing that $c(t) \leq H(t)$ (and equivalently, $\dot{H}(t) \geq 0$) is an indicator of prudent behavior.[5] In particular, real wealth equivalent income at the national level is equal to consumption if consumption is constant.

Unfortunately, as discussed by Asheim (1997) and Sefton and Weale (2006), this approach is not compatible with the property that real income at the national level equals net national product in a closed economy with a stationary technology. To see this, note that by Sefton and Weale (2006, Eq. 23),

$$Y(t) - H(t) = \int_t^\infty \left(1 - \frac{R_\infty(t)}{R_\infty(\tau)}\right)\frac{\pi(\tau)}{\pi(t)}\dot{c}(\tau)d\tau.$$

Hence, if consumption is always increasing and the real interest rate (and therefore the real infinitely long-term interest rate, too) is always decreasing, then $\dot{c}(\tau) > 0$ and $1 - R_\infty(t)/R_\infty(\tau) < 0$, so that $Y(t) < H(t)$.

[5] Note that $H(t)$ is the maximal consumption that can be sustained indefinitely only if intertemporal redistribution of consumption can be made at relative prices $\{\pi(t)\}_{t=0}^\infty$ without changing these prices. Due to general equilibrium effects, this will not be the case at the national level.

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